

## ON THE CHARACTER SPACE OF BANACH VECTOR-VALUED FUNCTION ALGEBRAS

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**ABSTRACT.** Given a compact space  $X$  and a commutative Banach algebra  $A$ , the character spaces of  $A$ -valued function algebras on  $X$  are investigated. The class of natural  $A$ -valued function algebras, those whose characters can be described by means of characters of  $A$  and point evaluation homomorphisms, is introduced and studied. For an admissible Banach  $A$ -valued function algebra  $\mathcal{A}$  on  $X$ , conditions under which the character space  $\mathfrak{M}(\mathcal{A})$  is homeomorphic to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$  are presented, where  $\mathfrak{A} = C(X) \cap \mathcal{A}$  is the subalgebra of  $\mathcal{A}$  consisting of scalar-valued functions. An illustration of the results is given by some examples.

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### 1. Introduction and Preliminaries

Let  $A$  be a commutative unital Banach algebra over the complex field  $\mathbb{C}$ . Every nonzero homomorphism  $\phi : A \rightarrow \mathbb{C}$  is called a *character* of  $A$ . Denoted by  $\mathfrak{M}(A)$ , the set of all characters of  $A$  is nonempty and its elements are automatically continuous [13, Lemma 2.1.5]. Consider the Gelfand transform  $\hat{A} = \{\hat{a} : a \in A\}$ , where  $\hat{a} : \mathfrak{M}(A) \rightarrow \mathbb{C}$  is defined by  $\hat{a}(\phi) = \phi(a)$ . The *Gelfand topology* of  $\mathfrak{M}(A)$  is the weakest topology with respect to which every  $\hat{a} \in \hat{A}$  is continuous. Endowed with the Gelfand topology,  $\mathfrak{M}(A)$  is compact and Hausdorff. By [13, Theorem 2.1.8], an ideal  $M$  in  $A$  is maximal if and only if  $M = \ker \phi$ , for some  $\phi \in \mathfrak{M}(A)$ . For this reason, sometimes  $\mathfrak{M}(A)$  is called the *maximal ideal space* of  $A$ . For more on the theory of commutative Banach algebras see, for example, [5, 6, 13, 19].

**1.1. Function Algebras.** Let  $X$  be a compact Hausdorff space and  $C(X)$  be the Banach algebra of all continuous functions  $f : X \rightarrow \mathbb{C}$  equipped with the uniform norm  $\|f\|_X = \sup\{|f(x)| : x \in X\}$ . A subalgebra  $\mathfrak{A}$  of  $C(X)$  is called a *function algebra* on  $X$  if  $\mathfrak{A}$  separates the points of  $X$  and contains the constant functions. If  $\mathfrak{A}$  is equipped with some complete algebra norm  $\|\cdot\|$ , then  $\mathfrak{A}$  is called a *Banach function algebra*. If the norm  $\|\cdot\|$  of  $\mathfrak{A}$  is equivalent to the uniform norm  $\|\cdot\|_X$ , then  $\mathfrak{A}$  is called a *uniform algebra*.

Identifying the character space of a Banach function algebra  $\mathfrak{A}$  has been always a problem of interest for mathematicians in this field. For every

$x \in X$ , the evaluation homomorphism  $\varepsilon_x : f \mapsto f(x)$  is a character of  $\mathfrak{A}$ , and the mapping  $J : X \rightarrow \mathfrak{M}(\mathfrak{A})$ ,  $x \mapsto \varepsilon_x$ , imbeds  $X$  homeomorphically as a compact subset of  $\mathfrak{M}(\mathfrak{A})$ . If  $J$  is surjective, one calls  $\mathfrak{A}$  *natural* [6, Chapter 4]. In this case, the character space  $\mathfrak{M}(\mathfrak{A})$  is identical to  $X$ . Note that every semisimple commutative Banach algebra  $A$  can be considered, through its Gelfand representation, as a natural Banach function algebra on its character space  $\mathfrak{M}(A)$ .

A relation between the character space  $\mathfrak{M}(\mathfrak{A})$  of a Banach function algebra  $\mathfrak{A}$  and the character space  $\mathfrak{M}(\bar{\mathfrak{A}})$  of its uniform closure  $\bar{\mathfrak{A}}$  was revealed in [10] as follows.

**Theorem 1.1** (Honary [10]). *The restriction map  $\mathfrak{M}(\bar{\mathfrak{A}}) \rightarrow \mathfrak{M}(\mathfrak{A})$ ,  $\psi \mapsto \psi|_{\mathfrak{A}}$ , is a homeomorphism if and only if  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathfrak{A}$ .*

The above result appears to be very useful in identifying the character spaces in a wide class of Banach function algebras. We establish an analogue of this result for vector-valued function algebras in Section 3.

**1.2. Vector-valued Function Algebras.** Let  $A$  be a commutative unital Banach algebra, and let  $C(X, A)$  be the space of all  $A$ -valued continuous functions on  $X$ . Algebraic operations and the uniform norm  $\|\cdot\|_X$  on  $C(X, A)$  are defined in the obvious way.

**Definition 1.2** (c.f. [2, 15]). A subalgebra  $\mathcal{A}$  of  $C(X, A)$  is called an *A-valued function algebra* on  $X$  if (1)  $\mathcal{A}$  contains the constant functions  $X \rightarrow A$ ,  $x \mapsto a$ , for all  $a \in A$ , and (2)  $\mathcal{A}$  separates the points of  $X$  in the sense that, for every pair  $x, y \in X$  with  $x \neq y$ , and for every maximal ideal  $M$  of  $A$ , there exists some  $f \in \mathcal{A}$  such that  $f(x) - f(y) \notin M$ . If  $\mathcal{A}$  is endowed with some algebra norm  $\|\cdot\|$  such that the restriction of  $\|\cdot\|$  to  $A$  is equivalent to the original norm of  $A$  and  $\|f\|_X \leq \|f\|$ , for every  $f \in \mathcal{A}$ , then  $\mathcal{A}$  is called a *normed A-valued function algebra* on  $X$ . If the given norm is complete, then  $\mathcal{A}$  is called a *Banach A-valued function algebra*. If the given norm is equivalent to the uniform norm  $\|\cdot\|_X$ , then  $\mathcal{A}$  is called an *A-valued uniform algebra*. When no confusion can arise, we use the same notation  $\|\cdot\|$  for the norm of  $\mathcal{A}$ .

Continuing the work of Yood [18], Hausner [9] proved that  $\tau$  is a character of  $C(X, A)$  if, and only if, there exist a point  $x \in X$  and a character  $\phi \in \mathfrak{M}(A)$  such that  $\tau(f) = \phi(f(x))$ , for all  $f \in C(X, A)$ , whence  $\mathfrak{M}(C(X, A))$  is homeomorphic to  $X \times \mathfrak{M}(A)$ . (In this regard, see [1].) We call a Banach  $A$ -valued function algebra *natural* if, like  $C(X, A)$ , its character space is identical to  $X \times \mathfrak{M}(A)$ . For instance, in Example 4.1, we will see that the  $A$ -valued Lipschitz algebra  $\text{Lip}(X, A)$  is natural; see also [7], [11]. Natural  $A$ -valued function algebras are studied in Section 2.

**1.3. Notations and conventions.** Throughout,  $X$  is a compact Hausdorff space, and  $A$  is a *semisimple* commutative unital Banach algebra. The unit element of  $A$  is denoted by  $\mathbf{1}$ , and the set of invertible elements of  $A$  is

denoted by  $\text{Inv}(A)$ . If  $f : X \rightarrow \mathbb{C}$  is a function and  $a \in A$ , we write  $fa$  to denote the  $A$ -valued function  $X \rightarrow A$ ,  $x \mapsto f(x)a$ . If  $\mathfrak{A}$  is a function algebra on  $X$ , we let  $\mathfrak{A}A$  be the linear span of  $\{fa : f \in \mathfrak{A}, a \in A\}$ , so that any element  $f \in \mathfrak{A}A$  is of the form  $f = f_1a_1 + \cdots + f_na_n$  with  $f_j \in \mathfrak{A}$  and  $a_j \in A$ . Given an element  $a \in A$ , we use the same notation  $a$  for the constant function  $X \rightarrow A$  given by  $a(x) = a$ , for all  $x \in X$ , and consider  $A$  as a closed subalgebra of  $C(X, A)$ . Since  $A$  has a unit element  $\mathbf{1}$ , we identify  $\mathbb{C}$  with the closed subalgebra  $\mathbb{C}\mathbf{1}$  of  $A$ . Whence every continuous function  $f : X \rightarrow \mathbb{C}$  can be considered as the continuous  $A$ -valued function  $f\mathbf{1} : x \mapsto f(x)\mathbf{1}$ . We drop  $\mathbf{1}$  using the same notation  $f$  for this  $A$ -valued function and adopt the identification  $C(X) = C(X)\mathbf{1}$  as a closed subalgebra of  $C(X, A)$ . Finally, for a family  $\mathcal{M}$  of  $A$ -valued functions on  $X$ , a point  $x \in X$ , and a character  $\phi \in \mathfrak{M}(A)$ , we set

$$\mathcal{M}(x) = \{f(x) : f \in \mathcal{M}\}, \quad \phi[\mathcal{M}] = \{\phi \circ f : f \in \mathcal{M}\}.$$

## 2. NATURAL VECTOR-VALUED FUNCTION ALGEBRAS

Let  $\mathcal{A}$  be an  $A$ -valued function algebra on  $X$ . Assume that  $M$  is a maximal ideal of  $A$ ,  $x_0 \in X$ , and set

$$(2.1) \quad \mathcal{M} = \{f \in \mathcal{A} : f(x_0) \in M\}.$$

The fact that  $\mathcal{M}$  is an ideal of  $\mathcal{A}$  is obvious. We prove that  $\mathcal{M}$  is maximal. Take a function  $g \in \mathcal{A} \setminus \mathcal{M}$  so that  $g(x_0) \notin M$ . Since  $M$  is maximal in  $A$ , there exist  $a \in M$  and  $b \in A$  such that  $\mathbf{1} = a + g(x_0)b$ . Consider  $b$  as a constant function of  $X$  into  $A$  and let  $f = \mathbf{1} - gb$ . Then  $f(x_0) = a \in M$  so that  $f \in \mathcal{M}$  and  $\mathbf{1} = f + gb$  which means that the ideal of  $\mathcal{A}$  generated by  $\mathcal{M} \cup \{g\}$  is equal to  $\mathcal{A}$ . Hence  $\mathcal{M}$  is maximal in  $\mathcal{A}$ .

**Definition 2.1.** An  $A$ -valued function algebra  $\mathcal{A}$  on  $X$  is called *natural* on  $X$ , if every maximal ideal  $\mathcal{M}$  of  $\mathcal{A}$  is of the form (2.1), for some  $x_0 \in X$  and  $M \in \mathfrak{M}(A)$ .

In case  $A = \mathbb{C}$ , natural  $A$ -valued function algebras coincide with natural (complex) function algebras.

**Theorem 2.2.** Let  $\mathcal{A}$  be an  $A$ -valued function algebra on  $X$ . If  $\mathcal{M}$  is a maximal ideal in  $\mathcal{A}$  and  $\mathcal{M}(x_0) \neq A$ , for some  $x_0 \in X$ , then

- (1)  $\mathcal{M}(x_0)$  is a maximal ideal of  $A$ ;
- (2)  $\mathcal{M}(x) = A$  for  $x \neq x_0$ ;
- (3)  $\mathcal{M} = \{f \in \mathcal{A} : f(x_0) \in M\}$ , where  $M = \mathcal{M}(x_0)$ .

*Proof.* It is easily verified that  $\mathcal{M}(x_0)$  is an ideal. We show that  $\mathcal{M}(x_0)$  is maximal. Assume that  $a \notin \mathcal{M}(x_0)$ . Then  $a$ , as a constant function on  $X$ , does not belong to  $\mathcal{M}$ . Hence, the ideal of  $\mathcal{A}$  generated by  $\mathcal{M} \cup \{a\}$  is equal to  $\mathcal{A}$  meaning that  $\mathbf{1} = f + ag$ , for some  $f \in \mathcal{M}$  and  $g \in \mathcal{A}$ . In particular,  $\mathbf{1} = f(x_0) + ag(x_0)$  which implies that the ideal of  $A$  generated by  $\mathcal{M}(x_0) \cup \{a\}$  is equal to  $A$ . Hence,  $\mathcal{M}(x_0)$  is maximal.

Now, assume that  $x \neq x_0$ . Since  $\mathcal{A}$  separates the points of  $X$  (Definition 1.2), for the maximal ideal  $\mathcal{M}(x_0)$  in  $A$ , there is a function  $f \in \mathcal{A}$  such that  $f(x) - f(x_0) \notin \mathcal{M}(x_0)$ . Define  $g(s) = f(s) - f(x)$  so that  $g(x_0) \notin \mathcal{M}(x_0)$ . This implies that  $g \notin \mathcal{M}$ . Since  $\mathcal{M}$  is maximal, there are  $h \in \mathcal{M}$  and  $k \in \mathcal{A}$  such that  $h + kg = \mathbf{1}$ . Hence,  $\mathbf{1} = h(x) \in \mathcal{M}(x)$  and  $\mathcal{M}(x) = A$ .  $\square$

It is proved in [1] that the algebra  $C(X, A)$  satisfies all conditions in Theorem 2.2. Therefore  $C(X, A)$  is natural.

**Corollary 2.3.** *Let  $\mathcal{A}$  be an  $A$ -valued function algebra on  $X$ .*

- (1) *The algebra  $\mathcal{A}$  is natural if, and only if, for every proper ideal  $\mathcal{I}$  in  $\mathcal{A}$ , there exists some  $x_0 \in X$  such that  $\mathcal{I}(x_0) \neq A$ .*
- (2) *If  $\mathcal{I}$  is an ideal in  $\mathcal{A}$  such that  $\mathcal{I}(x_0)$  and  $\mathcal{I}(x_1)$ , for  $x_0 \neq x_1$ , are proper ideals in  $A$ , then  $\mathcal{I}$  cannot be maximal in  $\mathcal{A}$ .*

The next discussion requires a concept of zero sets. The zero set of a function  $f : X \rightarrow A$  is defined as  $Z(f) = \{x : f(x) = \mathbf{0}\}$ . This concept of zero set, however, is not useful here in our discussion because, in general, the algebra  $A$  may contain nonzero singular elements. Instead, the following slightly modified version of this concept appears to be very useful.

**Definition 2.4.** For a function  $f : X \rightarrow A$ , the *singular set* of  $f$  is defined to be

$$(2.2) \quad Z_s(f) = \{x \in X : f(x) \notin \text{Inv}(A)\}.$$

The following is an analogy of [6, Proposition 4.1.5 (i)].

**Theorem 2.5.** *Let  $\mathcal{A}$  be a Banach  $A$ -valued function algebra on  $X$ . Then  $\mathcal{A}$  is natural if, and only if, for each finite set  $\{f_1, \dots, f_n\}$  of elements in  $\mathcal{A}$  with  $\bigcap_{j=1}^n Z_s(f_j) = \emptyset$ , there exist  $g_1, \dots, g_n \in \mathcal{A}$  such that*

$$f_1 g_1 + \dots + f_n g_n = \mathbf{1}.$$

*Proof.* ( $\Rightarrow$ ) Suppose that  $\mathcal{A}$  is natural and, for a finite set  $\{f_1, \dots, f_n\}$  in  $\mathcal{A}$ , assume that  $Z_s(f_1) \cap \dots \cap Z_s(f_n) = \emptyset$ . Let  $\mathcal{I}$  be the ideal generated by  $\{f_1, \dots, f_n\}$ . If  $\mathcal{I} \neq \mathcal{A}$ , then, since  $\mathcal{A}$  is natural, by Corollary 2.3, there exists a point  $x_0 \in X$  such that  $\mathcal{I}(x_0) \neq A$ . In particular, the elements  $f_1(x_0), \dots, f_n(x_0)$  are all singular in  $A$ , which means that  $x_0 \in Z_s(f_1) \cap \dots \cap Z_s(f_n)$ , a contradiction. Therefore,  $\mathcal{I} = \mathcal{A}$  whence there exist  $g_1, \dots, g_n \in \mathcal{A}$  such that  $f_1 g_1 + \dots + f_n g_n = \mathbf{1}$ .

( $\Leftarrow$ ) To show that  $\mathcal{A}$  is natural, we take a maximal ideal  $\mathcal{M}$  of  $\mathcal{A}$  and, using Corollary 2.3, we show that  $\mathcal{M}(x_0) \neq A$ , for some  $x_0 \in X$ . Assume, towards a contradiction, that, for every  $x \in X$ , there exists a function  $f_x \in \mathcal{M}$  such that  $f_x(x) = \mathbf{1}$ . Set  $V_x = f_x^{-1}(\text{Inv}(A))$ . Then  $\{V_x : x \in X\}$  is an open covering of the compact space  $X$ . So there exist finitely many points  $x_1, \dots, x_n \in X$  such that  $X \subset V_{x_1} \cup \dots \cup V_{x_n}$ . Then  $Z_s(f_{x_1}) \cap \dots \cap Z_s(f_{x_n}) = \emptyset$ . By the assumption, there exist functions  $g_1, \dots, g_n \in \mathcal{A}$  such that  $f_{x_1} g_1 + \dots + f_{x_n} g_n = \mathbf{1}$ . Hence,  $\mathbf{1} \in \mathcal{M}$ , which is a contradiction.  $\square$

Let  $f \in \mathcal{A}$  and suppose that  $Z_s(f) = \emptyset$  so that  $f(X) \subset \text{Inv}(A)$ . Since the inverse mapping  $a \mapsto a^{-1}$  of  $\text{Inv}(A)$  onto itself is continuous, the mapping  $x \mapsto f(x)^{-1}$ , denoted by  $\mathbf{1}/f$ , is a continuous  $A$ -valued function on  $X$ . Hence  $f$  is invertible in  $C(X, A)$ . However,  $f$  may not be invertible in  $\mathcal{A}$ . Let us call  $\mathcal{A}$  a *full subalgebra* of  $C(X, A)$  if every  $f \in \mathcal{A}$  that is invertible in  $C(X, A)$  is invertible in  $\mathcal{A}$ . The following is an analogy of [3, Theorem 2.1].

**Theorem 2.6.** *Let  $\mathcal{A}$  be a Banach  $A$ -valued function algebra on  $X$  such that  $\bar{\mathcal{A}}$ , the uniform closure of  $\mathcal{A}$ , is natural. If  $\mathbf{1}/f \in \mathcal{A}$  whenever  $f \in \mathcal{A}$  and  $Z_s(f) = \emptyset$ , then  $\mathcal{A}$  is natural.*

*Proof.* We apply Theorem 2.5 to prove that  $\mathcal{A}$  is natural. Let  $f_1, \dots, f_n$  be elements in  $\mathcal{A}$  such that  $Z_s(f_1) \cap \dots \cap Z_s(f_n) = \emptyset$ . We prove the existence of a finite set  $\{g_1, \dots, g_n\}$  of elements in  $\mathcal{A}$  such that  $f_1 g_1 + \dots + f_n g_n = \mathbf{1}$ . Regarding  $f_1, \dots, f_n$  as elements of  $\bar{\mathcal{A}}$ , since  $\bar{\mathcal{A}}$  is natural, again by Theorem 2.5, there exist  $h_1, \dots, h_n$  in  $\bar{\mathcal{A}}$  such that  $f_1 h_1 + \dots + f_n h_n = \mathbf{1}$ . For each  $h_j$ , there is some  $g_j \in \mathcal{A}$  such that  $\|h_j - g_j\|_X < \sum_{j=1}^n \|f_j\|_X$ . Thus

$$(2.3) \quad \left\| \mathbf{1} - \sum_{j=1}^n f_j g_j \right\|_X = \left\| \sum_{j=1}^n f_j h_j - \sum_{j=1}^n f_j g_j \right\|_X \leq \sum_{j=1}^n \|f_j\|_X \|h_j - g_j\|_X < 1.$$

Hence, for every  $x \in X$ ,  $f(x) = \sum f_j(x) g_j(x)$  is an invertible element of  $A$ , so that for the function  $f = \sum f_j g_j$ , which belongs to  $\mathcal{A}$ , we have  $Z_s(f) = \emptyset$ . By the assumption, there is a function  $g$  in  $\mathcal{A}$  such that  $\mathbf{1} = fg = \sum f_j (g_j g)$ . Now, Theorem 2.5 shows that  $\mathcal{A}$  is natural.  $\square$

An application of the above theorem is given in Example 4.1.

Let  $\mathcal{A}$  be a Banach  $A$ -valued function algebra. For every point  $x \in X$  and character  $\phi \in \mathfrak{M}(A)$  define

$$\varepsilon_x \diamond \phi : \mathcal{A} \rightarrow \mathbb{C}, \quad \varepsilon_x \diamond \phi(f) = \varepsilon_x(\phi \circ f) = \phi(f(x)).$$

Then  $\varepsilon_x \diamond \phi$  is a character of  $\mathcal{A}$  with  $\ker(\varepsilon_x \diamond \phi) = \{f \in \mathcal{A} : f(x) \in \ker \phi\}$ , which of course is of the form (2.1). Define

$$(2.4) \quad \mathcal{J} : X \times \mathfrak{M}(A) \rightarrow \mathfrak{M}(\mathcal{A}), \quad (x, \phi) \rightarrow \varepsilon_x \diamond \phi.$$

**Theorem 2.7.** *The mapping  $\mathcal{J}$  is a homeomorphism of  $X \times \mathfrak{M}(A)$  onto a compact subset of  $\mathfrak{M}(\mathcal{A})$ . If  $\mathcal{A}$  is natural, then  $\mathfrak{M}(\mathcal{A})$  is homeomorphic to  $X \times \mathfrak{M}(A)$ .*

*Proof.* Take  $x \in X$ ,  $\phi \in \mathfrak{M}(A)$ , and set  $\tau_0 = \varepsilon_x \diamond \phi$ . Let  $W$  be a neighbourhood of  $\tau_0$  in  $\mathfrak{M}(\mathcal{A})$  of the form

$$W = \{\tau \in \mathfrak{M}(\mathcal{A}) : |\tau(f_i) - \tau_0(f_i)| < \varepsilon, 1 \leq i \leq n\},$$

where  $f_1, \dots, f_n \in \mathcal{A}$ . Take

$$U = \{y \in X : \|f_i(y) - f_i(x)\| < \varepsilon/2, 1 \leq i \leq n\},$$

$$V = \{\psi \in \mathfrak{M}(A) : |\psi(f_i(x)) - \phi(f_i(x))| < \varepsilon/2, 1 \leq i \leq n\}.$$

Then  $U$  is a neighbourhood of  $x$  in  $X$  and  $V$  is a neighbourhood of  $\phi$  in  $\mathfrak{M}(A)$ , so that  $U \times V$  is a neighbourhood of  $(x, \phi)$  in  $X \times \mathfrak{M}(A)$ . If  $(y, \psi) \in U \times V$  then, for every  $i$  ( $1 \leq i \leq n$ ),

$$\begin{aligned} |\psi(f_i(y)) - \phi(f_i(x))| &\leq |\psi(f_i(y)) - \psi(f_i(x))| + |\psi(f_i(x)) - \phi(f_i(x))| \\ &< \|\psi\| \|f_i(y) - f_i(x)\| + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This shows that  $\varepsilon_y \diamond \psi \in W$  and thus  $\mathcal{J}$  is continuous. Finally, if  $\mathcal{A}$  is natural then every maximal ideal of  $\mathcal{A}$  is of the form (2.1) which means that every character  $\tau \in \mathfrak{M}(\mathcal{A})$  is of the form  $\tau = \varepsilon_x \diamond \phi$ , for some  $x \in X$  and  $\phi \in \mathfrak{M}(A)$ . Hence,  $\mathcal{J}$  is a surjection and thus a homeomorphism.  $\square$

### 3. CHARACTERS ON VECTOR-VALUED FUNCTION ALGEBRAS

We turn to a more general case where a vector-valued function algebra may not be natural. Let  $\mathcal{A}$  be a Banach  $A$ -valued function algebra. We show that, under certain conditions, the character space  $\mathfrak{M}(\mathcal{A})$  is identical to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ , where  $\mathfrak{A} = C(X) \cap \mathcal{A}$  is the subalgebra of  $\mathcal{A}$  consisting of scalar-valued functions. To this end, we should restrict ourself to the class of admissible algebras. If  $f \in \mathcal{A}$  and  $\phi \in \mathfrak{M}(A)$ , it is clear that  $\phi \circ f \in C(X)$ ; it is not, however, clear whether the  $A$ -valued function  $(\phi \circ f)\mathbf{1}$  belongs to  $\mathcal{A}$ . In fact, [2, Example 2.4] shows that it may very well happen that  $(\phi \circ f)\mathbf{1} \notin \mathcal{A}$ .

**Definition 3.1** ([2]). The  $A$ -valued function algebra  $\mathcal{A}$  is called *admissible* if

$$(3.1) \quad \{(\phi \circ f)\mathbf{1} : f \in \mathcal{A}, \phi \in \mathfrak{M}(A)\} \subset \mathcal{A}.$$

Note that  $\mathcal{A}$  is admissible if, and only if,  $\phi[\mathcal{A}]\mathbf{1} \subset \mathcal{A}$ , for all  $\phi \in \mathfrak{M}(A)$ .

Admissible vector-valued function algebras exist around in abundant. Some typical examples are  $C(X, A)$ ,  $\text{Lip}(X, A)$ ,  $P(K, A)$ ,  $R(K, A)$ , etc. Tensor products of the form  $\mathfrak{A} \otimes A$ , where  $\mathfrak{A}$  is a (Banach) function algebra on  $X$ , can be seen as admissible  $A$ -valued function algebras. (More details are given in Example 4.4.)

During this section, we assume that  $\mathcal{A}$  is admissible and set  $\mathfrak{A} = \mathcal{A} \cap C(X)$ . Then  $\mathfrak{A}$  is the subalgebra of  $\mathcal{A}$  consisting of all complex functions in  $\mathcal{A}$ , it forms a complex function algebra by itself, and  $\mathfrak{A} = \phi[\mathcal{A}]$ , for all  $\phi \in \mathfrak{M}(A)$ . Our aim is to give a description of maximal ideals in  $\mathcal{A}$ . To begin, take a character  $\phi \in \mathfrak{M}(A)$  and a maximal ideal  $\mathbf{M}$  of  $\mathfrak{A}$ , and set

$$(3.2) \quad \mathcal{M} = \{f \in \mathcal{A} : \phi \circ f \in \mathbf{M}\}.$$

Then  $\mathcal{M}$  is a maximal ideal of  $\mathcal{A}$ . One way to see this (though it can be seen directly) is as follows. Take  $\psi \in \mathfrak{M}(\mathfrak{A})$  with  $\mathbf{M} = \ker \psi$  and define

$$\psi \diamond \phi : \mathcal{A} \rightarrow \mathbb{C}, \quad \psi \diamond \phi(f) = \psi(\phi \circ f).$$

Note that  $\psi(\phi \circ f)$  is meaningful since  $\phi \circ f \in \mathfrak{A}$ . The functional  $\psi \diamond \phi$  is a character of  $\mathcal{A}$  with  $\ker(\psi \diamond \phi) = \mathcal{M}$ . Hence  $\mathcal{M}$  is a maximal ideal of  $\mathcal{A}$ .

The main question is whether any maximal ideal  $\mathcal{M}$  of  $\mathcal{A}$  is of the form (3.2).

**Lemma 3.2.** *A maximal ideal  $\mathcal{M}$  of  $\mathcal{A}$  is of the form (3.2) if and only if  $\phi[\mathcal{M}] \neq \mathfrak{A}$  for some  $\phi \in \mathfrak{M}(A)$ .*

*Proof.* If  $\mathcal{M}$  is of the form (3.2) then clearly  $\phi[\mathcal{M}] \neq \mathfrak{A}$ . Conversely, assume that  $\phi[\mathcal{M}] \neq \mathfrak{A}$  for some  $\phi \in \mathfrak{M}(A)$ . Then  $\phi[\mathcal{M}]$  is an ideal of  $\mathfrak{A}$ . We show that it is maximal. If  $g \notin \phi[\mathcal{M}]$ , then  $g = g\mathbf{1}$  (as an  $A$ -valued function on  $X$ ) does not belong to  $\mathcal{M}$ . Since  $\mathcal{M}$  is maximal in  $\mathcal{A}$ , the ideal generated by  $\mathcal{M} \cup \{g\}$  is equal to  $\mathcal{A}$ . This implies that  $\mathbf{1} = f + gh$ , for some  $f \in \mathcal{M}$  and  $h \in \mathcal{A}$ . Since  $\phi \circ g = g$ , we get  $\mathbf{1} = \phi \circ f + g(\phi \circ h)$ . This means that the ideal of  $\mathfrak{A}$  generated by  $\phi[\mathcal{M}] \cup \{g\}$  is equal to  $\mathfrak{A}$ . Thus,  $\phi[\mathcal{M}]$  is maximal. Set  $\mathbf{M} = \phi[\mathcal{M}]$  and  $\mathcal{M}_1 = \{f \in \mathcal{A} : \phi \circ f \in \mathbf{M}\}$ . Then  $\mathcal{M} \subset \mathcal{M}_1$  and both  $\mathcal{M}$  and  $\mathcal{M}_1$  are maximal ideals. Hence,  $\mathcal{M} = \mathcal{M}_1$ .  $\square$

If  $\mathcal{M} = \ker \tau$ , for some  $\tau \in \mathfrak{M}(\mathcal{A})$ , then  $\mathcal{M}$  is of the form (3.2) if and only if  $\tau = \psi \diamond \phi$ , for some  $\psi \in \mathfrak{M}(\mathfrak{A})$  and  $\phi \in \mathfrak{M}(A)$ . Let us extend the mapping  $\mathcal{J}$  in (2.4) to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$  as follows.

$$(3.3) \quad \mathcal{J} : \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A) \rightarrow \mathfrak{M}(\mathcal{A}), \quad \mathcal{J}(\psi, \phi) = \psi \diamond \phi.$$

The mapping is injective for if  $\psi \diamond \phi = \psi' \diamond \phi'$  then

$$\begin{aligned} \phi(a) &= \psi(\phi(a)) = \psi'(\phi'(a)) = \phi'(a) \quad (a \in A), \\ \psi(f) &= \psi(\phi(f)) = \psi'(\phi'(f)) = \psi'(f) \quad (f \in \mathfrak{A}), \end{aligned}$$

which implies that  $\phi = \phi'$  and  $\psi = \psi'$ . The main question is whether  $\mathcal{J}$  is surjective. If  $\tau \in \mathfrak{M}(\mathcal{A})$  then  $\phi = \tau|_A \in \mathfrak{M}(A)$  and  $\psi = \tau|_{\mathfrak{A}} \in \mathfrak{M}(\mathfrak{A})$ . The question is whether the equality  $\tau = \psi \diamond \phi$  holds true; of course, it does hold if  $\phi[\mathcal{M}] \neq \mathfrak{A}$ .

**Theorem 3.3.** *If the mapping  $\mathcal{J}$  in (3.3) is a surjection, then it is a homeomorphism and, therefore, the character space  $\mathfrak{M}(\mathcal{A})$  is identical to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ .*

*Proof.* Suppose that  $\mathcal{J}$  is a surjection (and thus a bijection). Since both the domain and the range are compact Hausdorff spaces, it suffices to prove that  $\mathcal{J}$  is open. Take  $\psi_0 \in \mathfrak{M}(\mathfrak{A})$ ,  $\phi_0 \in \mathfrak{M}(A)$  and set  $\tau_0 = \mathcal{J}(\psi_0, \phi_0) = \psi_0 \diamond \phi_0$ . Let  $U$  and  $V$  be neighborhoods of  $\psi_0$  and  $\phi_0$  of the following form

$$\begin{aligned} U &= \{\psi \in \mathfrak{M}(\mathfrak{A}) : |\psi(f) - \psi_0(f)| < \varepsilon_1 \quad (f \in F_1)\}, \\ V &= \{\phi \in \mathfrak{M}(A) : |\phi(a) - \phi_0(a)| < \varepsilon_2 \quad (a \in F_2)\}, \end{aligned}$$

where  $F_1$  and  $F_2$  are finite sets in  $\mathfrak{A}$  and  $A$ , respectively. Take  $F = F_1 \cup F_2$  as a finite set in  $\mathcal{A}$ ,  $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$  and set

$$W = \{\tau \in \mathfrak{M}(\mathcal{A}) : |\tau(f) - \tau_0(f)| < \varepsilon \quad (f \in F)\}.$$

Then  $W$  is a neighborhood of  $\tau_0$  in  $\mathfrak{M}(\mathcal{A})$  and  $\mathcal{J}(U \times V) \subset W$ . Hence  $\mathcal{J}$  is open.  $\square$



The rest of this section is devoted to investigating conditions under which  $\mathcal{J}$  is surjective.

**Theorem 3.4** ( $\mathcal{P}$ ). *For a character  $\tau \in \mathfrak{M}(\mathcal{A})$  with  $\mathcal{M} = \ker \tau$  and  $\phi = \tau|_A$ , the following are equivalent.*

- (i)  $\phi[\mathcal{M}] \neq \mathfrak{A}$ .
- (ii)  $\mathcal{M}$  is of the form (3.2) with  $\mathbf{M} = \phi[\mathcal{M}]$ .
- (iii) For every  $f \in \mathcal{A}$ , if  $\phi \circ f = \mathbf{0}$  then  $f \in \mathcal{M}$ .
- (iv) For every  $f \in \mathcal{A}$ ,  $\tau(\phi \circ f) = \tau(f)$ .
- (v) For every  $f \in \mathcal{A}$ , if  $f(X) \subset \mathcal{M}$  then  $f \in \mathcal{M}$ .
- (vi)  $\tau = \psi \diamond \phi$ , for some  $\psi \in \mathfrak{M}(\mathfrak{A})$ .

*Proof.* The equivalence (i)  $\Leftrightarrow$  (ii) is just Lemma 3.2. The implication (ii)  $\Rightarrow$  (iii) is clear. To see the implication (iii)  $\Rightarrow$  (iv), let  $g = f - (\phi \circ f)\mathbf{1}$ . Then  $\phi \circ g = \mathbf{0}$  and thus  $g \in \mathcal{M}$  and  $\tau(g) = 0$ . Hence,  $\tau(\phi \circ f) = \tau(f)$ . The implication (iv)  $\Rightarrow$  (iii) is clear.

To prove (iii)  $\Leftrightarrow$  (v), we note that  $f(X) \subset \mathcal{M}$  if and only if  $\phi \circ f = \mathbf{0}$ . In fact,  $f(X) \subset \mathcal{M}$  means that, for every  $x \in X$ , the element  $f(x)$ , as a constant function of  $X$  into  $A$ , belongs to  $\mathcal{M}$ . This, in turn, means that  $\tau(f(x)) = \phi(f(x)) = 0$ , for all  $x \in X$ , which means that  $\phi \circ f = \mathbf{0}$ .

To prove (iii)  $\Rightarrow$  (vi), first note that  $\mathcal{A}$  being admissible implies that

$$\mathfrak{A} = \phi[\mathcal{A}] = \{\phi \circ f : f \in \mathcal{A}\}.$$

Define  $\psi : \mathfrak{A} \rightarrow \mathbb{C}$  by  $\psi(\phi \circ f) = \tau(f)$ . This is well-defined for if  $\phi \circ f = \phi \circ g$  then, by the assumption,  $f - g \in \mathcal{M}$  which in turn implies that  $\tau(f) = \tau(g)$ . Obviously,  $\psi \in \mathfrak{M}(\mathfrak{A})$  and  $\tau = \psi \diamond \phi$ .

Finally, we prove that (vi)  $\Rightarrow$  (i). Towards a contradiction, assume that  $\phi[\mathcal{M}] = \mathfrak{A}$ . Then  $\phi \circ f = \mathbf{1}$ , for some  $f \in \mathcal{M}$ . Hence  $1 = \psi(\mathbf{1}) = \psi(\phi \circ f) = \tau(f) = 0$  which is absurd.  $\square$

*Convention.* We say that ' $\mathcal{A}$  has property  $\mathcal{P}$ ' if every  $\mathcal{M} \in \mathfrak{M}(\mathcal{A})$  satisfies one (and hence all) of conditions in Theorem 3.4. Hence  $\mathcal{A}$  has  $\mathcal{P}$  if and only if the mapping  $\mathcal{J}$  in (3.3) is surjective.

Let  $\bar{\mathcal{A}}$  denote the uniform closure of  $\mathcal{A}$  in  $C(X, A)$ . The restriction map

$$(3.4) \quad \mathfrak{M}(\bar{\mathcal{A}}) \rightarrow \mathfrak{M}(\mathcal{A}), \quad \bar{\tau} \mapsto \bar{\tau}|_{\mathcal{A}},$$

is one-to-one and continuous with respect to the Gelfand topology [10]. We write  $\mathfrak{M}(\bar{\mathcal{A}}) = \mathfrak{M}(\mathcal{A})$  if it is a homeomorphism.

**Proposition 3.5.** *If  $\mathcal{A}$  has  $\mathcal{P}$  then  $\bar{\mathcal{A}}$  has  $\mathcal{P}$ . If  $\bar{\mathcal{A}}$  has  $\mathcal{P}$  and  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathcal{A}$ , then  $\mathcal{A}$  has  $\mathcal{P}$ .*

*Proof.* Suppose that  $\mathcal{A}$  has  $\mathcal{P}$ . Take  $\bar{\tau} \in \mathfrak{M}(\bar{\mathcal{A}})$ , set  $\tau = \bar{\tau}|_{\mathcal{A}}$  and  $\phi = \bar{\tau}|_A = \tau|_A$ . Since  $\mathcal{A}$  has  $\mathcal{P}$ , by Theorem 3.4 (iv),  $\tau(\phi \circ f) = \tau(f)$ , for all  $f \in \mathcal{A}$ . Given  $f \in \bar{\mathcal{A}}$ , there is a sequence  $\{f_n\}$  in  $\mathcal{A}$  such that  $\|f_n - f\|_X \rightarrow 0$ . Hence,  $\|\phi \circ f_n - \phi \circ f\|_X \rightarrow 0$ , and thus

$$\bar{\tau}(\phi \circ f) = \lim_{n \rightarrow \infty} \bar{\tau}(\phi \circ f_n) = \lim_{n \rightarrow \infty} \tau(\phi \circ f_n) = \lim_{n \rightarrow \infty} \tau(f_n) = \lim_{n \rightarrow \infty} \bar{\tau}(f_n) = \bar{\tau}(f).$$



Again, by Theorem 3.4 (iv), we see that  $\mathcal{A}$  has  $\mathcal{P}$ .

Now, assume that  $\mathcal{A}$  has  $\mathcal{P}$ , and  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathcal{A}$ . Take  $\tau \in \mathfrak{M}(\mathcal{A})$  and  $\phi = \tau|_A$ . Extend  $\tau$  to a character  $\bar{\tau} : \mathcal{A} \rightarrow \mathbb{C}$  (this is possible since  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathcal{A}$ ). Note that still we have  $\phi = \bar{\tau}|_A$ . Since  $\mathcal{A}$  satisfies  $\mathcal{P}$ , we have  $\bar{\tau}(\phi \circ f) = \bar{\tau}(f)$ , for all  $f \in \mathcal{A}$ . This implies that  $\tau(\phi \circ f) = \tau(f)$ , for all  $f \in \mathcal{A}$ , and thus  $\mathcal{A}$  has  $\mathcal{P}$ .  $\square$

The following is a vector-valued version of Theorem 1.1.

**Theorem 3.6.** *For an admissible Banach  $A$ -valued function algebra  $\mathcal{A}$  with  $\mathfrak{A} = C(X) \cap \mathcal{A}$ , let  $\bar{\mathcal{A}}$  and  $\bar{\mathfrak{A}}$  be the uniform closures of  $\mathcal{A}$  and  $\mathfrak{A}$ , respectively. Consider the following statements:*

- (i)  $\mathfrak{M}(\bar{\mathcal{A}}) = \mathfrak{M}(\mathcal{A})$ .
- (ii)  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathcal{A}$ .
- (iii)  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathfrak{A}$ .
- (iv)  $\mathfrak{M}(\bar{\mathfrak{A}}) = \mathfrak{M}(\mathfrak{A})$ .

Then (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iii)  $\Leftrightarrow$  (iv). If  $\mathcal{A}$  satisfies  $\mathcal{P}$ , then (iii)  $\Rightarrow$  (ii).

*Proof.* The equivalences (i)  $\Leftrightarrow$  (ii) and (iii)  $\Leftrightarrow$  (iv) follow from the main theorem in [10]. The implication (ii)  $\Rightarrow$  (iii) is obvious, because  $\mathfrak{A} \subset \mathcal{A}$ .

Assume that  $\mathcal{A}$  satisfies  $\mathcal{P}$ , and  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathfrak{A}$ . Fix a function  $f \in \mathcal{A}$  and take an arbitrary character  $\tau \in \mathfrak{M}(\mathcal{A})$ . Since  $\mathcal{A}$  has  $\mathcal{P}$ , we have  $\tau = \psi \circ \phi$ , where  $\psi = \tau|_{\mathfrak{A}}$  and  $\phi = \tau|_A$ . Since  $\phi \circ f \in \mathfrak{A}$ , we have

$$|\tau(f)| = |\psi(\phi \circ f)| \leq \|\widehat{\phi \circ f}\| \leq \|\phi \circ f\|_X \leq \|f\|_X.$$

Hence  $\|\hat{f}\| \leq \|f\|_X$ , for all  $f \in \mathcal{A}$ .  $\square$

#### 4. EXAMPLES

To illustrate the results, we devote this section to some examples.

**Example 4.1.** Let  $(X, \rho)$  be a compact metric space. A function  $f : X \rightarrow A$  is called an  $A$ -valued Lipschitz function if

$$(4.1) \quad L(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X, x \neq y \right\} < \infty.$$

The space of  $A$ -valued Lipschitz functions on  $X$  is denoted by  $\text{Lip}(X, A)$ . For any  $f \in \text{Lip}(X, A)$ , the Lipschitz norm of  $f$  is defined by  $\|f\|_L = \|f\|_X + L(f)$ . This makes  $\text{Lip}(X, A)$  an admissible Banach  $A$ -valued function algebra on  $X$  with  $\text{Lip}(X) = \text{Lip}(X, A) \cap C(X)$ , where  $\text{Lip}(X) = \text{Lip}(X, \mathbb{C})$  is the classical complex Lipschitz algebra on  $X$ .

The algebra  $\text{Lip}(X)$  satisfies all conditions in the Stone-Weierstrass Theorem and thus it is dense in  $C(X)$ . On the other hand, by [9, Lemma 1],  $C(X)A$  is dense in  $C(X, A)$  and thus  $\text{Lip}(X)A$  is dense in  $C(X, A)$ . Since  $\text{Lip}(X, A)$  contains  $\text{Lip}(X)A$ , we see that  $\text{Lip}(X, A)$  is dense in  $C(X, A)$ .

It is easy to verify that if  $f \in \text{Lip}(X, A)$  and  $Z_s(f) = \emptyset$ , then  $1/f \in \text{Lip}(X, A)$ . Since  $C(X, A)$  is natural, Theorem 2.6 now implies that  $\text{Lip}(X, A)$

is natural. By Theorem 2.7,  $\mathfrak{M}(\text{Lip}(X, A))$  is homeomorphic to  $X \times \mathfrak{M}(A)$ . See also [7] and [11].

**Example 4.2.** Assume that  $A = \mathbb{C}^n$ , for some positive integer  $n$ . Then, for every admissible Banach  $A$ -valued function algebra  $\mathcal{A}$  on  $X$ , the mapping  $\mathcal{J}$  in (3.3) is surjective and thus  $\mathfrak{M}(\mathcal{A})$  is identical to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(\mathbb{C}^n)$ .

To see this, we show that  $\mathcal{A}$  satisfies condition (i) of Theorem 3.4. Note that  $\mathfrak{M}(\mathbb{C}^n) = \{\pi_1, \dots, \pi_n\}$ , where  $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}$  is the projection on  $i$ -th component. Assume  $\mathcal{M}$  is an ideal in  $\mathcal{A}$  and  $\mathbf{1} \in \pi_i[\mathcal{M}]$ , for all  $i = 1, \dots, n$ . Hence, for every  $i$ , there is some  $f_i \in \mathcal{M}$  such that  $\pi_i \circ f_i = \mathbf{1}$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{C}^n$ . Then  $e_i$ , as a constant function of  $X$  into  $A$ , belongs to  $\mathcal{A}$ . Since  $\mathcal{M}$  is an ideal, we have  $\mathbf{1} = e_1 f_1 + \dots + e_n f_n \in \mathcal{M}$ . Hence,  $\mathcal{M} = \mathcal{A}$  and  $\mathcal{M}$  cannot be maximal.

If  $\mathcal{X} = \{1, \dots, n\}$ , then  $\mathbb{C}^n = C(\mathcal{X})$ . The above example states that, given any admissible Banach  $C(\mathcal{X})$ -valued function algebra, we have  $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(C(\mathcal{X}))$ . If  $\mathcal{X}$  is an arbitrary compact Hausdorff space, it is unknown whether the result still holds for any admissible Banach  $C(\mathcal{X})$ -valued function algebra. But, the following shows that it does hold for admissible  $C(\mathcal{X})$ -valued uniform algebras.

**Example 4.3.** Assume that  $A = C(\mathcal{X})$ , for some compact Hausdorff space  $\mathcal{X}$ . Then, for every admissible  $A$ -valued uniform algebra  $\mathcal{A}$  on  $X$ , the mapping  $\mathcal{J}$  in (3.3) is surjective, and thus  $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(C(\mathcal{X}))$ .

To see this, first we show that  $\mathcal{A}$  is isometrically isomorphic to  $C(\mathcal{X}, \mathfrak{A})$ . Take a function  $f \in \mathcal{A}$ . Then  $f(x)$ , for every  $x \in X$ , is a function in  $C(\mathcal{X})$ . Define  $\tilde{f} : \mathcal{X} \rightarrow \mathfrak{A}$  by  $\tilde{f}(\xi)(x) = f(x)(\xi)$ . In fact,  $\tilde{f}(\xi) = \phi_\xi \circ f$  where  $\phi_\xi$  is the evaluation character of  $A = C(\mathcal{X})$  at  $\xi$ , and, since  $\mathcal{A}$  is admissible,  $\tilde{f}(\xi)$  belongs to  $\mathfrak{A}$ . Now, define  $T : \mathcal{A} \rightarrow C(\mathcal{X}, \mathfrak{A})$  by  $Tf = \tilde{f}$ . It is easily verified that  $T$  is an algebra homomorphism, and

$$\|f\|_X = \sup_{x \in X} \|f(x)\| = \sup_{x \in X} \sup_{\xi \in \mathcal{X}} |f(x)(\xi)| = \sup_{\xi \in \mathcal{X}} \|\tilde{f}(\xi)\| = \|\tilde{f}\|_{\mathcal{X}}.$$

Since the range of  $T$  contains all elements of the form  $g_1 h_1 + \dots + g_n h_n$ , where  $n \in \mathbb{N}$ ,  $g_i \in C(\mathcal{X})$  and  $h_i \in \mathfrak{A}$ , and these functions are dense in  $C(\mathcal{X}, \mathfrak{A})$ , we have  $T$  surjective. It follows that  $T$  is an isometric isomorphism. By [9, Theorem],  $\mathfrak{M}(C(\mathcal{X}, \mathfrak{A}))$  is identical to  $\mathfrak{M}(\mathfrak{A}) \times \mathcal{X}$ , which means that  $\mathfrak{M}(\mathcal{A})$  is identical to  $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ .

**Example 4.4 (Tensor Products).** Let  $\mathfrak{A}$  be a Banach function algebra on  $X$  and consider the algebraic tensor product  $\mathfrak{A} \otimes A$ . There exists, by [5, Theorem 42.6], a linear operator  $T : \mathfrak{A} \otimes A \rightarrow \mathfrak{A}A$  such that

$$(4.2) \quad T\left(\sum_{i=1}^n f_i \otimes a_i\right) = \sum_{i=1}^n f_i a_i.$$

The operator  $T$  is an algebra isomorphism so that  $\mathfrak{A} \otimes A$  can be seen as an admissible  $A$ -valued function algebra on  $X$ . We identify every element

$f \in \mathfrak{A} \otimes A$  with its image  $Tf$  as an  $A$ -valued function on  $X$ . Let  $\|\cdot\|_\gamma$  be an algebra cross-norm on  $\mathfrak{A} \otimes A$  so that the completion  $\mathfrak{A} \widehat{\otimes}_\gamma A$  is a Banach algebra. The mapping  $T$  extends to an isometric isomorphism of  $\mathfrak{A} \widehat{\otimes}_\gamma A$  onto a Banach  $A$ -valued function algebra on  $X$ . For example, if  $\|\cdot\|_\epsilon$  is the injective tensor norm, then  $\mathfrak{A} \widehat{\otimes}_\epsilon A$  is isometrically isomorphic to the uniform closure  $\overline{\mathfrak{A}A}$  of  $\mathfrak{A}A$  and  $\|f\|_\epsilon = \|f\|_X$ , for all  $f \in \mathfrak{A} \otimes A$ .

It is proved in [4] that

- (1)  $\mathfrak{A} \widehat{\otimes}_\gamma A$  is an admissible Banach  $A$ -valued function algebra on  $X$ .
- (2) If  $f \in \mathfrak{A} \widehat{\otimes}_\gamma A$  and  $\phi \in A^*$  then  $\phi \circ f \in \mathfrak{A}$  and  $\|\phi \circ f\| \leq \|\phi\| \|f\|_\gamma$ .

We now show that every  $\tau \in \mathfrak{M}(\mathfrak{A} \widehat{\otimes}_\gamma A)$  is of the form  $\tau = \psi \diamond \phi$ , with  $\phi = \tau|_A$  and  $\psi = \tau|_{\mathfrak{A}}$ . Since  $\mathfrak{A} \otimes A$  is dense in  $\mathfrak{A} \widehat{\otimes}_\gamma A$ , it is enough to show that  $\tau = \psi \diamond \phi$  on  $\mathfrak{A} \otimes A$ . First, note that every  $f \in \mathfrak{A} \otimes A$  can be seen, through the isomorphism (4.2), as  $f = f_1 a_1 + \cdots + f_n a_n$ . Hence,  $\phi \circ f = \phi(a_1) f_1 + \cdots + \phi(a_n) f_n$  and

$$\begin{aligned} \tau(f) &= \tau(f_1 a_1 + \cdots + f_n a_n) \\ &= \tau(f_1) \tau(a_1) + \cdots + \tau(f_n) \tau(a_n) \\ &= \psi(f_1) \phi(a_1) + \cdots + \psi(f_n) \phi(a_n) \\ &= \psi(\phi(a_1) f_1 + \cdots + \phi(a_n) f_n) \\ &= \psi(\phi \circ f). \end{aligned}$$

This proves that  $\tau = \psi \diamond \phi$  on  $\mathfrak{A} \otimes A$  and thus  $\tau = \psi \diamond \phi$  on  $\mathfrak{A} \widehat{\otimes}_\gamma A$ . We conclude that  $\mathfrak{M}(\mathfrak{A} \widehat{\otimes}_\gamma A) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ . This result, however, can be derived from the following more general result due to Tomiyama [17].

**Theorem 4.5** ([17]). *Suppose that  $A$  and  $B$  are commutative Banach algebras. If  $A \widehat{\otimes}_\gamma B$  is a Banach algebra for a cross-norm  $\gamma$ , then  $\mathfrak{M}(A \widehat{\otimes}_\gamma B)$  is homeomorphic to  $\mathfrak{M}(A) \times \mathfrak{M}(B)$ .*

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